Sketchy solutions of Midterm, Hedva2.ME

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(1) a. Denote $c_n = \frac{x^n}{a^{\sqrt{n}}}$. To check whether $\sum c_n$ converges we use e.g. Cauchy criterion. $\sqrt[n]{|c_n|} = \frac{|x|}{a^{\frac{1}{\sqrt{n}}}} \to |x|$. Therefore the series converges absolutely for $|x| < 1$ and diverges when $|x| > 1$.

The boundary points. Suppose $|x| = 1$ and $a > 1$. Then $|c_n| < \frac{1}{n^2}$ for $n \gg 1$. (For example take ln of both sides.) Therefore the series converges absolutely. Suppose $|x| = 1$ and $0 < a \leq 1$. Then $|c_n| \geq 1$, in particular $c_n \not\to 0$. Thus the series diverges.

b. Apply Cauchy criterion: $\sqrt[n]{|a_n|} = \left(\frac{n-1}{n+1}\right)^{n-1} \to e^{-2} < 1$. Therefore the series converges absolutely.

(2) a. Note that the function is not defined for $x = 0 = z$. The level surfaces are: $\frac{x^2+y^2}{x^2+z^2} = c$. Note that the whole expression is non-negative, so $c \geq 0$.

- If $c = 0$ then $x = 0 = y$, but $z \neq 0$, i.e. this is a line without a point.
- Suppose $c > 0$ then the equation can be written in the form: $x^2(1-c) + y^2 = cz^2$. If $c < 1$ this is a cone around $\hat{z}$-axis (with the point $(0,0,0)$ erased). If $c = 1$ this is the union of two planes: $y = z$ and $y = -z$. (with the point $(0,0,0)$ erased) If $c > 1$ this is again a cone, $y^2 = cz^2 + (c-1)x^2$, around $\hat{y}$-axis (with the point $(0,0,0)$ erased).

b. Let the equation of the plane be $ax+by+cz = d$. Checking the point F one gets: $d = 6c$. Then the point P gives: $a = c$. Finally the point Q gives $b = 4c$. So the plane is: $x + 4y + z = 6$. The line through the two points is parameterized by $(0,1,3) + t(1,2,-3)$. This intersects the plane for $t = -\frac{1}{6}$. Thus the intersection point is: $(-\frac{1}{6}, \frac{2}{3}, \frac{7}{2})$.

(3) a. The function is continuous and $f|_{(x,0)} = x$, $f|_{(0,y)} = y$. Therefore $\partial_x f|_{(0,0)} = 1 = \partial_y f|_{(0,0)}$. For the differentiability it remains to check the remainder: $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt[3]{x^3+y^3} - 1\cdot x - 1\cdot y}{\sqrt{x^2+y^2}}$.

We check this limit along some paths. For example, along $x = 0$ (or $y = 0$) the limit is 0. But along $x = y$ the expression is $\frac{(\sqrt[3]{2}-2)x}{\sqrt{2}|x|}$, thus the limit does not exist. Therefore $f(x,y)$ is not differentiable at $(0,0)$.

b.

(1)

$$(\partial_u\phi + \partial_v\phi + \partial_w\phi)|_{(1,2,3)} = \Big(\partial_x f \cdot \partial_u(u^2 - v^2) + \partial_y f \cdot \partial_u(u^2 vw)\Big)|_{(1,2,3)} + \Big(\partial_x f \cdot \partial_v(u^2 - v^2) + \partial_y f \cdot \partial_v(u^2 vw)\Big)|_{(1,2,3)} +$$

$$+ \Big(\partial_x f \cdot \partial_w(u^2 - v^2) + \partial_y f \cdot \partial_w(u^2 vw)\Big)|_{(1,2,3)} = \partial_x f 2(u-v)|_{(1,2,3)} + \partial_y f(2uvw + u^2 w + u^2 v)|_{(1,2,3)} = 4 + 17$$